Received: *February 9, 2001*, Accepted: *April 17, 2001*

Non-abelian vacua in $D = 5$, $N = 4$ gauged supergravity

Ali H. Chamseddine

Center for Advanced Mathematical Sciences (CAMS) and Physics Department
Americal University of Beirut, Lebanon
E-mail: chams@aub.edu.lb

Mikhail S. Volkov*

Institute for Theoretical Physics, Friedrich Schiller University of Jena
Max-Wien Platz 1, D-07743 Jena, Germany
E-mail: vol@tpi.uni-jena.de

Abstract: We study essentially non-abelian backgrounds in the five dimensional $N = 4$ gauged SU(2) $\times$ U(1) supergravity. Static configurations that are invariant under either the SO(4) spatial rotations or with respect to the SO(3) rotations and translations along the fourth spatial coordinate are considered. By analyzing consistency conditions for the equations for supersymmetric Killing spinors we derive the Bogomol'nyi equations and obtain their globally regular solutions. The SO(4) symmetric configurations contain the purely magnetic non-abelian fields together with the purely electric abelian field and possess two unbroken supersymmetries. The SO(3) configurations have only the non-abelian fields and preserve four supersymmetries.

Keywords: Superstring Vacua, AdS-CFT Correspondance.

*Supported by the DFG grant Wi 777/4-2.

JHEP04(2001)023

Contents

1. Introduction **1**

2. The $D = 5$, $N = 4$ gauged supergravity **2**

3. Solutions with SO(4) symmetry **3**
- 3.1 Supersymmetry constraints 5
- 3.2 Bogomol'nyi equations 8

4. Solutions with SO(3) symmetry **9**

5. Concluding remarks **12**

1. Introduction

The gauged supergravities in five dimensions have been recently the subject of intensive research in view of the AdS/CFT correspondence (see [1] for a review) as well as in connection with the brane world scenario [10]. It is believed that solutions in such models provide the dual supergravity description for flat space gauge theories. This has inspired the widespread interest in such solutions, but only configurations with abelian gauge fields have been studied so far. At the same time, the bulk theories generically contain Yang-Mills fields, which of course have nothing to do with the non-abelian fields of the dual gauge theories but rather give rise to non-trivial warp factors in the ten-dimensional metric. It would therefore be interesting to obtain supergravity solutions with non-trivial Yang-Mills fields in the bulk and implement them in the context of the bulk/boundary correspondence.

Some results in this direction have been obtained in four dimensions. In [2] the non-abelian monopole-type supersymmetric vacua were found in the context of the $N = 4$ half-gauged $\mathrm{SU}(2) \times (\mathrm{U}(1))^3$ supergravity of Freedman and Schwarz [6], and their ten-dimensional analogs were obtained in [3]. It has been argued [8] that these solutions provide the dual supergravity description for the N=1 super-Yang-Mills theory. The non-abelian euclidean supersymmetric backgrounds and their ten-dimensional analogs were obtained in [13, 12], but the corresponding dual flat space theory has not been identified so far. Other known solution in D=4 can be related to reductions of heterotic string theory; see [5] and references therein. The only known non-abelian vacua in $D = 5$ are the heterotic solitons of [5], and also the BPS solutions with non-abelian matter [4].

In the present paper we study non-abelian supersymmetric backgrounds in five dimensions in the context of $N = 4$ SU(2)×U(1) gauged supergravity of Romans [11]. We consider static configurations that are invariant either under the SO(4) spatial rotations or with respect to the SO(3) rotations plus translations along the fourth spatial coordinate. By analyzing the consistency conditions for supersymmetric Killing spinors we derive the Bogomol'nyi equations and obtain their globally regular solutions. In the SO(4) case the configurations contain the purely magnetic non-abelian fields plus the purely electric abelian field and preserve only two unbroken supersymmetries out of sixteen. The SO(3) configurations have only the non-abelian fields and preserve four supersymmetries.

2. The $D = 5$, $N = 4$ gauged supergravity

The five dimensional $N = 4$ gauged SU(2)×$U(1)$ supergravity contains in the bosonic sector the gravitational field $\mathbf{g}_{\mu\nu}$, the SU(2) non-abelian gauge field A^a_μ ($a = 1, 2, 3$), the abelian gauge field a_μ, a pair of 2-form fields, and the dilaton ϕ [11]. Since the 2-forms are self-dual, one can set them to zero on shell, and then one can set the U(1) gauge coupling constant to zero, such that the model becomes ungauged in the U(1) sector. After a suitable rescaling of the fields one can set the SU(2) gauge coupling constant to one, and then the bosonic part of the action becomes

$$S = \int \biggl(-\frac{R}{4} + \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4}\,\eta^2 F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4\eta^4}\,f_{\mu\nu}f^{\mu\nu} - \\ - \frac{1}{4\sqrt{\mathbf{g}}}\,\varepsilon^{\mu\nu\rho\sigma\tau}F^a_{\mu\nu}F^a_{\rho\sigma}a_\tau + \frac{1}{8\eta^2}\biggr)\sqrt{\mathbf{g}}\,d^5x\,. \tag{2.1}$$

Here $\eta = \exp(\sqrt{\tfrac{2}{3}}\,\phi)$, also $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon_{abc}A^b_\mu A^c_\nu$, while the abelian field strength is $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$.

In the fermionic sector the theory contains four gravitini ψ^I_μ and four gaugini χ^I; we shall always omit the index $I = 1, \ldots, 4$ in what follows. One can set the fermions to zero on shell, however their SUSY variation in general do not vanish. To write down these variations, let us introduce 4×4 spacetime gamma matrices $\gamma^A = (\gamma^0, \gamma^r, \gamma^a)$ subject to

$$\gamma^A\gamma^B + \gamma^B\gamma^A = 2\eta^{AB}\,, \tag{2.2}$$

with $\eta_{AB} = (+, -, -, -, -)$, and also 4×4 matrices $\Gamma_j = (\Gamma_a, \Gamma_4, \Gamma_5)$ acting on the internal indices of the spinors and spanning the five-dimensional euclidean Clifford algebra

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2\delta_{ij}\,. \tag{2.3}$$

Notice that we decompose the five-dimensional tangent space indices as $(0, r, a)$, where r takes only one value, 'r', whereas $a = 1, 2, 3$. Introducing four sets of Pauli matrices: $\underline{\sigma}^a$, σ_b, $\underline{\tau}^a$, and τ_b, where matrices from different sets commute, for example $[\underline{\sigma}^a, \sigma_b] = 0$

$$\gamma^0 = \underline{\sigma}^3 \otimes 1\!\!1_2\,, \qquad \gamma^r = i\underline{\sigma}^1 \otimes 1\!\!1_2\,, \qquad \gamma^a = i\underline{\sigma}^2 \otimes \sigma_a\,, \tag{2.4}$$

and also

$$\Gamma_a = \underline{\tau}^2 \otimes \tau_a\,, \qquad \Gamma_4 = \underline{\tau}^1 \otimes 1\!\!1_2\,, \qquad \Gamma_5 = \underline{\tau}^3 \otimes 1\!\!1_2\,. \tag{2.5}$$

We shall not write explicitly the $\otimes$ symbol and the factors of $1\!\!1_2$ in what follows. One has $\Gamma_{i\ldots j} = \Gamma_{[i}, \ldots, \Gamma_{j]}$, similarly for products of γ^A. Introducing the 1-form basis $\Theta^A = \Theta^A_{\ \mu} dx^\mu$ such that $\mathbf{g}_{AB}\Theta^A\Theta^B$, the corresponding spin connection is $\omega^A_{\ B} = \omega^A_{\ B,\,C}\Theta^C$. The dual vector basis is defined by $E_A = E_A^{\ \mu}\,\partial_\mu$ so that the supercovariant derivative acting on the spinor supersymmetry parameter ϵ becomes

$$D_A\varepsilon = \left(E_A^{\ \mu}\frac{\partial}{\partial x^\mu} + \frac{1}{4}\,\omega_{CB,A}\gamma^{CB} + \frac{1}{2}\,A^a_A\Gamma_{a45}\right)\epsilon\,. \tag{2.6}$$

As a result, the linearized SUSY variations of the fermions in the model are given by [11]

$$\delta\psi_A = D_A\epsilon + \frac{1}{6\sqrt{2}\eta}\,\gamma_A\Gamma_{45}\,\epsilon - \frac{1}{6\sqrt{2}}\,(\gamma_A^{\ BC} - 4\delta^B_A\gamma^B)\left(\eta F^a_{BC}\Gamma_a + \frac{1}{\sqrt{2}\eta^2}\,f_{BC}\right)\epsilon\,, \tag{2.7}$$

$$\delta\chi = \frac{1}{\sqrt{2}}\,\gamma^A(E_A^{\ \mu}\partial_\mu\phi)\epsilon + \frac{1}{2\sqrt{6}\eta}\,\Gamma_{45}\,\epsilon - \frac{1}{2\sqrt{6}}\,\gamma^{AB}\left(\eta F^a_{AB}\Gamma_a - \frac{\sqrt{2}}{\eta^2}\,f_{AB}\right)\epsilon\,. \tag{2.8}$$

3. Solutions with SO(4) symmetry

Our first task is to consider fields which are static and invariant under the action of the SO(4) spatial symmetry group. The static, SO(4)-invariant spacetime metric can be represented in the curvature coordinates as

$$ds^2 = \mathrm{e}^{2\nu(r)}dt^2 - \frac{dr^2}{N(r)} - r^2 d\Omega_3^2\,, \tag{3.1}$$

where $d\Omega_3^2$ is the round metric of S^3. Introducing on S^3 the left-invariant forms θ^a subject to the Maurer-Cartan equation

$$d\theta^a + \varepsilon_{abc}\,\theta^b \wedge \theta^c = 0\,, \tag{3.2}$$

one has $d\Omega_3^2 = \theta^a\theta^a$. The static gauge field $A^a = A^a_\mu dx^\mu$ that is invariant under the combined action of the SO(4) spatial rotations and SU(2) gauge transformations is given by

$$A^a = (w(r) + 1)\,\theta^a\,, \tag{3.3}$$

the corresponding field strength being 'purely magnetic'

$$F^a = dw\wedge\theta^a + \frac{1}{2}\,(w^2-1)\,\varepsilon_{abc}\,\theta^b\wedge\theta^c\,. \tag{3.4}$$

We choose the abelian field to be 'purely electric'

$$f = Q(r)\,dt\wedge dr\,. \tag{3.5}$$

Finally, the dilaton is chosen as $\phi=\phi(r)$. As a result, all fields are expressed in terms of five functions ν, N, w, Q, ϕ of the radial coordinate r.

Varying the action (2.1) gives the second order lagrangian field equations. These admit important first integrals. When the five-metric splits into the direct sum ${}^{(5)}\mathbf{g} = \mathbf{g}_{00}\oplus{}^{(4)}\mathbf{g}$, one can check that not only in the SO(4)-symmetric case but also for arbitrary static fields, the field equations require that ${}^{(4)}\nabla(\ln\mathbf{g}_{00} - 2\sqrt{2/3}\,\phi) = 0$. Here ${}^{(4)}\nabla$ is the covariant laplacian with respect to ${}^{(4)}\mathbf{g}$. This implies that the following metric-dilaton relation can be imposed on shell:

$$\nu = \sqrt{\frac{2}{3}}\,(\phi-\phi_0)\,, \tag{3.6}$$

where ϕ_0 is an integration constant.

Next, the equations for the abelian field f

$$\nabla_\nu(\xi^{-4}f^{\nu\mu}) = \frac{1}{4\sqrt{\mathbf{g}}}\,\varepsilon^{\mu\nu\rho\sigma\tau}F^a_{\nu\rho}F^a_{\sigma\tau} \tag{3.7}$$

have the total derivative structure. In the SO(4)-symmetric case they can be integrated to give

$$Q = \frac{\mathrm{e}^{5\nu}}{\sqrt{N}r^3}\,(2w^3-6w+H)\,, \tag{3.8}$$

with H being integration constant. The remaining independent lagrangian equations read

$$\begin{aligned}
\frac{r^3}{2}N' + r^2(N-1) + r^2N\mathrm{e}^{2\nu}w'^2 + \mathrm{e}^{2\nu}(w^2-1)^2 +&\\
+\frac{r^4}{2}N\nu'^2 - \frac{r^4}{12}\mathrm{e}^{-2\nu} + \frac{1}{3}r^4N\mathrm{e}^{-6\nu}Q^2 &= 0\,,\\
\frac{r^3}{2}N' + 2r^2N\mathrm{e}^{2\nu}w'^2 + r^4N\nu'^2 - r^3N\nu' &= 0\,,\\
r^2Nw'' + \left(3r^2N\nu' + rN + \frac{r^2}{2}N'\right)w' - 2r\mathrm{e}^{-3\nu}\sqrt{N}(w^2-1)Q &= 2(w^2-1)w\,.
\end{aligned} \tag{3.9}$$

There is also an equation containing ν'', but it can be related to the equations above by virtue of the Bianchi identities.

3.1 Supersymmetry constraints

Our aim now is to study constraints imposed by supersymmetry. These can be expressed as a system of linear differential equations for the spinor supersymmetry parameters, $\delta\psi_A = \delta\chi = 0$. These equations are generically inconsistent, however one can find the consistency conditions, which can be represented as a set of non-linear first order differential equations for the background variables; see eqs. (3.29). These equations, usually called Bogomol'nyi equations, are further first integrals for the lagrangian field equations.

Let us introduce the 1-form basis

$$\Theta^0 = \mathrm{e}^{\nu} dt\,, \qquad \Theta^r = \frac{dr}{\sqrt{N}}\,, \qquad \Theta^a = r\,\theta^a\,, \tag{3.10}$$

such that the spacetime metric is

$$ds^2 = (\Theta^0)^2 - (\Theta^r)^2 - \delta_{ab}\,\Theta^a\Theta^b\,. \tag{3.11}$$

The dual vielbein vectors E_A are

$$E_0 = \mathrm{e}^{-\nu}\frac{\partial}{\partial t}\,, \qquad E_r = \sqrt{N}\frac{\partial}{\partial r}\,, \qquad E_a = \frac{1}{r}\,e_a\,. \tag{3.12}$$

Here θ^a are the left-invariant Maurer-Cartan forms on S^3 subject to eq. (3.2), and e_b are the dual left-invariant vectors, $\langle\theta^a, e_b\rangle = \delta^a_b$. It is worth noting that e_a, together with the right-invariant vectors, $\tilde{e}_a$, give rise to the angular momentum operators $L_a = \frac{i}{2}e_a$ and $\tilde{L}_a = \frac{i}{2}\tilde{e}_a$ with the commutation relations

$$[L_a, L_b] = i\varepsilon_{abc}\,L_c\,, \qquad [\tilde{L}_a, \tilde{L}_b] = i\varepsilon_{abc}\,\tilde{L}_c\,, \qquad [L_a, \tilde{L}_b] = 0\,. \tag{3.13}$$

One also has $L_aL_a = \tilde{L}_a\tilde{L}_a$. The spin-connection is given by

$$\omega_{AB,C} = \frac{1}{2}\left(C_{B,AC} + C_{C,AB} - C_{A,BC}\right),$$

where $C_{A,BC} = \eta_{AD}C^D_{\ BC}$ are determined by the commutation relations for the basis vectors of the vielbein, $[E_A, E_B] = C^C_{\ AB}\,E_C$. One finds the following non-zero components:

$$\omega_{0r,0} = \sqrt{N}\nu'\,, \qquad \omega_{ra,b} = \frac{\sqrt{N}}{r}\,\delta_{ab}\,, \qquad \omega_{ab,c} = \frac{1}{r}\,\varepsilon_{abc}\,. \tag{3.14}$$

Inserting the above expressions into (2.7), (2.8) and assuming that all spinors are time-independent, we compute the spinor SUSY variations $\delta\chi$ and $\delta\psi_A$. First, we obtain

$$\delta\chi = \sqrt{3}\,\underline{\sigma}^3\,\delta\psi_0 - \frac{i\sqrt{3}}{2r}\left(\nu - \sqrt{\frac{2}{3}}\phi\right)'\,\underline{\sigma}^1\,\epsilon\,, \tag{3.15}$$

which implies, in view of the metric-dilaton relation (3.6), that $\delta\chi$ is not an independent variation. We therefore focus on the gravitino variations $\delta\psi_A$:

$$
\begin{aligned}
\delta\psi_0 &= \left(-\frac{1}{2}\mathcal{A}_1\,\underline{\sigma}^2 - \frac{i}{2}\mathcal{A}\,\underline{\sigma}^3\underline{\tau}^2 + \frac{i}{6}\left(\mathcal{C} + \mathcal{B}\,\underline{\sigma}^3\,\right)\underline{\tau}^2\left(\sigma^a\tau^a\right) + i\mathcal{F}\,\underline{\sigma}^1\right)\epsilon\,,\\
\delta\psi_r &= \left(\sqrt{N}\frac{\partial}{\partial r} - \frac{1}{2}\mathcal{A}\,\underline{\sigma}^1\underline{\tau}^2 + \frac{1}{6}\left(2i\mathcal{C}\,\underline{\sigma}^2\, + \mathcal{B}\,\underline{\sigma}^1\,\right)\underline{\tau}^2\left(\sigma^a\tau^a\right) - \mathcal{F}\,\underline{\sigma}^3\right)\epsilon\,,\\
\delta\psi_a &= \Big(-\frac{2i}{r}\,L_a - \frac{i}{2}\,\mathcal{B}_1\,\underline{\sigma}^3\sigma_a - \frac{i}{2r}\left(\sigma_a + \tau_a\right) - \frac{i}{2}\,\mathcal{C}_1\tau_a - \\
&\quad - \frac{1}{2}\mathcal{A}\,\underline{\sigma}^2\underline{\tau}^2\sigma_a + \frac{1}{6}\left(\mathcal{B}\,\underline{\sigma}^2\Sigma_a\, - \mathcal{C}\,\underline{\sigma}^1\Lambda_a\right)\underline{\tau}^2 - \frac{i}{2}\,\mathcal{F}\,\sigma_a\Big)\epsilon\,.
\end{aligned}
\tag{3.16}
$$

Here $\Sigma_a = \tau_a + 2i\,\varepsilon_{abc}\sigma_b\tau_c$, $_c$, and also the following abbreviations have been introduced:

$$
\begin{aligned}
&\mathcal{A} = \frac{\mathrm{e}^{-\nu}}{3\sqrt{2}}\,, \qquad \mathcal{B} = \sqrt{2}\mathrm{e}^{\nu}\frac{w^2-1}{r^2}\,, \qquad \mathcal{C} = \frac{\sqrt{2N}}{r}\mathrm{e}^{\nu}w'\,, \\
&\mathcal{A}_1 = \sqrt{N}\nu'\,, \qquad \mathcal{B}_1 = \frac{\sqrt{N}}{r}\,, \qquad \mathcal{C}_1 = \frac{w}{r}\,, \qquad \mathcal{F} = \frac{\mathrm{e}^{2\nu}}{3r^3}\left(2w^3 - 6w + H\right).
\end{aligned}
\tag{3.17}
$$

The supersymmetry constraints are obtained by setting $\delta\psi_A = 0$, which gives the system of equations for the spinor ϵ. This spinor has 16 complex components subject to the symplectic Majorana condition, such that there are altogether 16 real independent components. Let us introduce two component spinors of four different types, $\underline{\psi}$, ψ, $\underline{\xi}$, ξ, that live in four spinor spaces where the operators $\underline{\sigma}^a$, σ_b, $\underline{\tau}^a$, and τ_b, respectively act. One can expand ϵ as

$$
\epsilon = \sum_{\alpha,\beta\,\gamma,\delta=\pm 1} C_{\alpha\beta\gamma\delta}\;\underline{\psi}^\alpha\otimes\psi^\beta\otimes\underline{\xi}^\gamma\otimes\xi^\delta\,,
\tag{3.18}
$$

where $C_{\alpha\beta\gamma\delta}$ are 16 functions of spacetime coordinates, and $\underline{\sigma}^3\underline{\psi}^\alpha = \alpha\underline{\psi}^\alpha$, also $\sigma^3\psi^\beta = \beta\psi^\beta$ and $\tau^3\xi^\delta = \delta\xi^\delta$, while $\underline{\xi}^\gamma$ are chosen to be eigenvectors of $\underline{\tau}^2$, $\underline{\tau}^2\underline{\xi}^\gamma = \gamma\underline{\xi}^\gamma$. The supersymmetry constraints $\delta\psi_A = 0$ is a system of $5\times16=80$ equations for 16 components of ϵ. Coefficients of this system, defined in (3.17), are determined by the underlying bosonic configuration. Although generically only the trivial solution is possible, one can find consistency conditions for the coefficients under which non-trivial solutions exist as well. The first step in doing this is to reduce somehow the size of the system. Since the underlying configuration is SO(4)-invariant, it is natural to consider the sector where ϵ is the eigenstate of the SO(4) angular momentum with zero eigenvalue(s).

Since SO(4) is locally isomorphic to the product of two copies of SO(3), the SO(4) angular momentum is essentially the sum of two SO(3) angular momenta. The two commuting SO(3) orbital angular momentum operators are given by eq. (3.13), but since the fermions also carry spin and isospin, we need the operator of the total angular momentum:

$$J_a = L_a + \frac{1}{2}\left(\sigma_a + \tau_a\right). \tag{3.19}$$

Since the commuting operators are J^2, J_3, $\tilde{L}^2$, $\tilde{L}_3$, there is a spinor ϵ annihilated by all these operators, such that $J_a\epsilon = \tilde{L}_a\epsilon$=0, and in view of the relation $L^2 = \tilde{L}^2$ one has also $L_a\epsilon = 0$, which implies that

$$L_a\epsilon = 0\,, \qquad (\sigma_a + \tau_a)\epsilon = 0\,. \tag{3.20}$$

The solution of these equations is

$$\epsilon = (\psi^{+1}\xi^{-1} - \psi^{-1}\xi^{+1}) \sum_{\alpha,\gamma=\pm 1} C_{\alpha\gamma}(r)\ \underline{\psi}^{\alpha}\underline{\xi}^{\gamma}\,, \tag{3.21}$$

and so we are now left with only four independent unknown functions $C_{\alpha\gamma}(r)$. From three matrices $\underline{\tau}^a$ only $\underline{\tau}^2$ enters the SUSY variations (3.16) and this leaves subspaces generated by $\underline{\xi}^{+1}$ and $\underline{\xi}^{-1}$ invariant. As a result, inserting (3.21) into (3.16) and denoting $\Psi_\gamma = \sum_{\alpha=\pm 1} C_{\alpha\gamma}(r)\underline{\psi}^{\alpha}$, the equations for Ψ_{+1} decouple from those for Ψ_{-1}. The conditions $\delta\psi_0 = 0$ and $\delta\psi_a = 0$ reduce then to

$$\left(\mathcal{A}_1\,\underline{\sigma}^3 - i\gamma\,\mathcal{A}\underline{\sigma}^2 + \gamma\,\mathcal{C}\underline{\sigma}^1 - i\gamma\,\mathcal{B}\underline{\sigma}^2 - 2\mathcal{F}\right)\Psi_\gamma = 0\,, \tag{3.22}$$

$$\left(\mathcal{B}_1\,\underline{\sigma}^3 - i\gamma\,\mathcal{A}\underline{\sigma}^2 - \mathcal{C}_1 + i\gamma\,\mathcal{B}\underline{\sigma}^2 + \mathcal{F}\right)\Psi_\gamma = 0\,, \tag{3.23}$$

while $\delta\psi_r = 0$ gives

$$\left(\sqrt{N}\frac{d}{dr} - \frac{\gamma}{2}\left(\mathcal{A} + \mathcal{B}\right)\underline{\sigma}^1 - i\gamma\,\mathcal{C}\,\underline{\sigma}^2 - \mathcal{F}\underline{\sigma}^3\right)\Psi_\gamma = 0\,. \tag{3.24}$$

Let us first consider eqs. (3.22), (3.23). For a given $\gamma = \pm 1$ these are four homogeneous algebraic equations for the two unknown quantities $C_{+1\gamma}$ and $C_{-1\gamma}$. A nontrivial solution exists if the 4×2 matrix of the system has rank 1, which gives three conditions on the coefficients of the matrix:

$$\begin{aligned}
\mathcal{A}_1^2 + \mathcal{C}^2 &= (\mathcal{A} + \mathcal{B})^2 + 4\mathcal{F}^2\,,\\
\mathcal{B}_1^2 &= (\mathcal{F} - \mathcal{C}_1)^2 + (\mathcal{A} - \mathcal{B})^2\,,\\
(\mathcal{A} - \mathcal{B})(\mathcal{C} - \mathcal{A} - \mathcal{B}) &= (\mathcal{A}_1 - 2\mathcal{F})(\mathcal{F} - \mathcal{B}_1 - \mathcal{C}_1)\,.
\end{aligned} \tag{3.25}$$

Notice that these relations do not contain γ. Under these conditions the algebraic equations (3.22), (3.23) become consistent and admit two solutions

$$C_{-1\gamma}(r) = \gamma\,\frac{2\mathcal{F} - \mathcal{A}_1}{\mathcal{C} - \mathcal{A} - \mathcal{B}}\,C_{+1\gamma}(r) \equiv \gamma\,\mathcal{Q}\,C_{+1\gamma}(r) \tag{3.26}$$

corresponding to two different values of γ.

Now, inserting these solutions into the differential constraints (3.24) gives two linear first order differential equations for one function $C_{+1+1}(r)$, and the same pair of equation arises for $C_{+1-1}(r)$. Since two differential equations for the same function must be compatible, this gives a further constraint on the coefficients:

$$\sqrt{N}\mathcal{Q}' + 2\mathcal{A}\mathcal{F} = \frac{\mathcal{A}+\mathcal{B}}{2} - \mathcal{C} - \left(\frac{\mathcal{A}+\mathcal{B}}{2} + \mathcal{C}\right)\mathcal{Q}^2 . \tag{3.27}$$

It turns out however that this new constraint is fulfilled by virtue of eqs. (3.25) (checking of which is not completely trivial). The differential equations can now be solved to give

$$C_{+1\gamma}(r) = C_\gamma \exp\left(\int^r \frac{dr}{\sqrt{N}}\left\{\mathcal{F} + \left(\frac{\mathcal{A}+\mathcal{B}}{2} + \mathcal{C}\right)\mathcal{Q}\right\}\right), \tag{3.28}$$

where C_γ are two integration constants. This finally gives two non-trivial solutions for supersymmetry Killing spinors. The consistency conditions for the existence of these solutions are given by eqs. (3.25).

3.2 Bogomol'nyi equations

Summarizing the results of the preceding subsection, eqs. (3.25) contain the complete set of consistency conditions under which non-trivial solutions for supersymmetry Killing spinors exist. These conditions can be represented as a system of first order Bogomol'nyi equations for the background variables:

$$\begin{aligned}
N &= \left(\frac{1}{3}\xi^2 V - w\right)^2 + 2\xi^2(w^2-1)^2 - \frac{2}{3}(w^2-1) + \frac{1}{18\xi^2}, \\
r\frac{dw}{dr} &= \frac{1}{6\xi^2 N}\left\{2V(1-w^2)\,\xi^4 + (H-4w^3)\,\xi^2 - w\right\}, \\
r\frac{d\xi}{dr} &= -\frac{\xi}{3N}\left\{V^2\xi^4 + (12\,(w^2-1)^2 - 4Vw)\,\xi^2 + w^2 + 2\right\},
\end{aligned} \tag{3.29}$$

with $\xi = \mathrm{e}^\nu/r$ and $V = 2w^3 - 6w + H$. One can directly check that these equations are compatible with the lagrangian equations of motion (3.9). Any solution to the Bogomol'nyi equations preserves *two* supersymmetries.

One can obtain some simple solutions. For example, setting $H = 0$, we find that $w = 0$ is a solution. The corresponding geometry

$$ds^2 = r_0^2\,\mathrm{e}^{1/12\xi^2}\left(\xi\,dt^2 - \frac{1}{8\xi^5}\,d\xi^2 - \frac{1}{\xi}\,d\Omega_3^2\right) \tag{3.30}$$

is singular both at the origin of the spherical coordinate system and at infinity (here r_0 is the integration constant).

The geometry of the solutions can be regular at the origin, $r = 0$, if only $H = 4$. Introducing the new variable $Y = 1/\xi^2 + 2w^2 + 4w - 2 - 4/w$, the Bogomol'nyi equations (3.29) reduce then to

$$w^2 Y\frac{dY}{dw} = 4\,(w-1)^2 Y + 16\,(w-1)(2w+1)(w+2)\,. \tag{3.31}$$

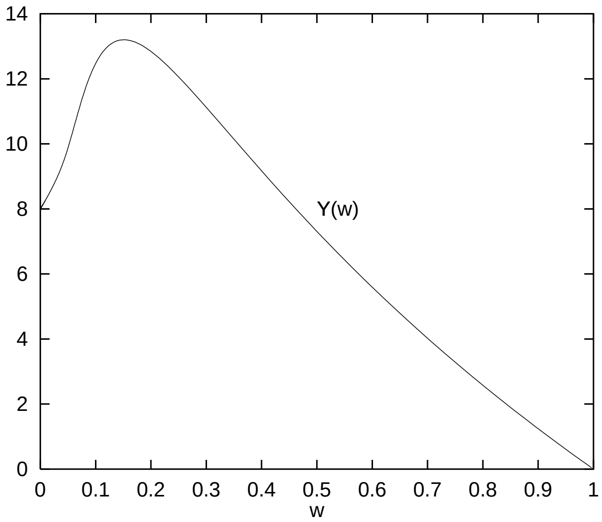

Figure 1: Solution to the Bogomol'nyi equation (3.31) with the boundary conditions specified by eq. (3.32).

Some solutions to this Abel's equation are known in a closed analytical form: $Y = 4(2w+1)(w-1)/w$ and $Y = -2(2w+1)(w+2)/w$, which however give rise to $\xi^2 < 0$. The numerical analysis on the other hand reveals a smooth solution with the following asymptotics (see figure 1):

$$\begin{aligned} Y &= 8 + 4\cdot 7\,w + 4\cdot 23\,w^2 + 8\cdot 89\,w^3 + 12\cdot 157\,w^4 + \cdots \text{as } w\to 0\,; \\ Y &= 12\,x + 4\,x^2 + 2\,x^3 + \frac{14}{15}\,x^4 + \frac{3}{10}\,x^5 - \frac{23}{210}\,x^6 + \cdots \quad \text{as } w \to 1\,, \end{aligned} \tag{3.32}$$

here $x = 1 - w$. The appearance of the prime numbers in these expansions suggests that the analytical solution with such asymptotics, if exists, should be sought for in a parametric form rather then as $Y(w)$. Passing to the $w(r)$, $N(r)$, $\nu(r)$ parameterization of this solution one finds that the geometry is globally regular; see figure 2. At the origin one has $w = 1 + O(r^2)$, $N = 1 + O(r^2)$, $\nu = O(r^2)$, such that the curvature is bounded and the gauge field vanishes as $r \to 0$. At infinity, $r \to \infty$, the leading behavior of the field amplitudes is $N \sim 1/w \sim r\mathrm{e}^{-\nu} \sim \ln r$, such that the geometry is not asymptotically flat (and not asymptotically AdS).

4. Solutions with SO(3) symmetry

Our next task is to consider static fields that are invariant under the SO(3) spatial rotations and in addition under translations along the fourth spatial direction. We parameterize the metric as

$$ds^2 = \mathrm{e}^{2\nu}\,dt^2 - \mathrm{e}^{2\tau}\left(\frac{d\tau^2}{N} + d\Omega_2^2\right) - \mathrm{e}^{2\mu}\,(dx^4)^2\,, \tag{4.1}$$

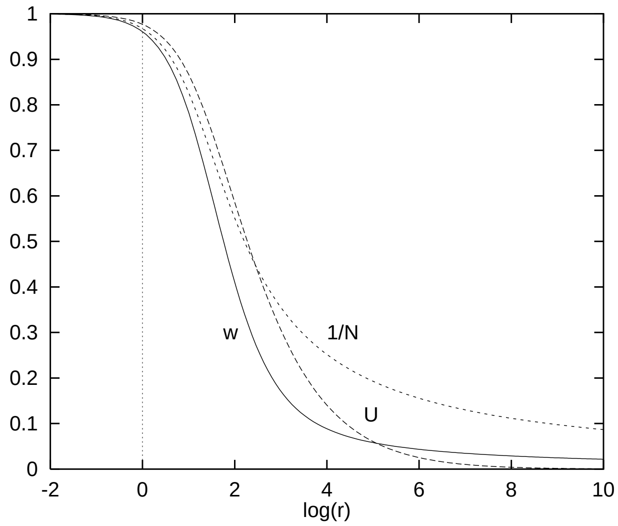

Figure 2: The same solution as in figure 1 parameterized by $w(r)$, $N(r)$, $\nu(r)$ such that eqs. (3.29) are fulfilled. Here U$\equiv \exp(-\nu)$.

where $d\Omega_2^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$, and choose the gauge field according to

$$A^a \mathbf{T}_a = w\,(-\mathbf{T}_2\, d\vartheta + \mathbf{T}_1\, \sin\vartheta\, d\varphi) + \mathbf{T}_3\, \cos\vartheta\, d\varphi\,. \tag{4.2}$$

Here ν, N, μ, w, and the dilaton ϕ depend only on τ, and $[\mathbf{T}_a, \mathbf{T}_b] = i\varepsilon_{abc}\mathbf{T}_c$ are the gauge group generators. This gauge field is 'purely magnetic', and moreover its field strength is such that $\varepsilon^{\mu\nu\rho\sigma\tau} F^a_{\mu\nu} F^a_{\rho\sigma} = 0$. As a result, the abelian vector field decouples, and we can set it to zero.

Our strategy is very much similar to the one described above for the SO(4)-symmetric fields. For this reason we shall mention only the essential points. First, it turns out that the lagrangian equations of motion allow us to impose on-shell the 'metric-dilaton relations'

$$\nu = \mu - \mu_0 = \sqrt{\frac{2}{3}}\,(\phi - \phi_0) \tag{4.3}$$

similar to the one in eq. (3.6), μ_0 and ϕ_0 being integration constants. The remaining independent equations read

$$\begin{aligned} \frac{3}{2}\,N' - 9N + 1 - 6N\,\frac{\xi'}{\xi} + 10\,N\xi^2 w'^2 + 2N\,\frac{\xi'^2}{\xi^2} + \frac{1}{2\xi^2} &= 0\,, \\ \frac{3}{2}\,N' - 1 + 2\xi^2(w^2-1)^2 + 6N\xi^2 w'^2 + N\,\frac{\xi'^2}{\xi^2} &= 0\,, \\ Nw'' + \left(\frac{N'}{2} + 3N + 4N\frac{\xi'}{\xi}\right) w' &= w^3 - w\,, \end{aligned} \tag{4.4}$$

with $' = d/(d\tau)$. The next step is to study the supersymmetry constraints $\delta\chi = \delta\psi_A = 0$ to derive the Bogomol'nyi equations. Let us split the tangent space indices as $A = (0, \tau, 2, 3, 4)$. It turns out that the metric-dilaton relation (4.3) implies that $\delta\psi_4$ and $\delta\chi$ fermionic SUSY variations are not independent but can be expressed in terms of $\delta\psi_0$ via a relation similar to the one in eq. (3.15). As a result, the independent supersymmetry constrains are $\delta\psi_0 = \delta\psi_2 = \delta\psi_3 = 0$, and also $\delta\psi_\tau = 0$, which gives a system of 64 equations.

In order to truncate the system, we require that $J_a\epsilon = 0$. Here $J_a = L_a + \frac{1}{2}(\sigma_a + \tau_a)$ is the total angular momentum with L_a being the usual SO(3) angular momentum acting on the ϑ, φ variables. Since now L^2 does not commute with J_a, we cannot require that ϵ is annihilated separately by the operators L_a and $\frac{1}{2}(\sigma_a + \tau_a)$, as was possible in the SO(4) case, but only by their sum. As a result, ϵ is constructed in terms of tensor products of eigenfunctions of L_3 and those of $\frac{1}{2}(\sigma_3 + \tau_3)$ with eigenvalues $0, \pm 1$. For more details we refer to [12] where a similar problem in four spacetime dimensions was considered.

The resulting ansatz for ϵ fixes the angular dependence of spinors and reduces the $\delta\psi_0 = \delta\psi_2 = \delta\psi_3 = 0$ constraints to a system of algebraic equations, whose consistency conditions are obtained similarly as was done above. These consistency conditions can be represented as a system of Bogomol'nyi equations,

$$\begin{aligned} N &= \frac{S^2}{18\xi^2 P}, \\ w' &= \frac{3w}{S}\left(1 + 2\xi^2(w^2 - 1)\right), \\ \xi' &= -\frac{6\xi^3}{S}\left(1 + w^2 + 2\xi^2(w^2 - 1)^2\right), \end{aligned} \tag{4.5}$$

with $S = 4(w^2 - 1)^2\xi^4 + 4(w^2 + 1)\xi^2 + 1$, $P = 8(w^2 - 1)^2\xi^4 + 6(w^2 + 1)\xi^2 + 1$, and $\xi = \exp(\nu - \tau)$. One can check that these Bogomol'nyi equations are compatible with the lagrangian equation (4.4). The remaining $\delta\psi_\tau = 0$ constraint equations turn out to be compatible with each other by virtue of eqs. (4.4), and they completely specify the τ -dependence of the spinors. This finally gives *four* independent supersymmetry Killing spinors.

Introducing $Y = 1/(2\xi^2)$ and $x = w^2$, the problem of solving the Bogomol'nyi equations (4.5) reduces to one equation

$$x(Y + x - 1)\frac{dY}{dx} + (x + 1)Y + (x - 1)^2 = 0\,. \tag{4.6}$$

For reasons that will be explained shortly, this equation exactly coincides with the one previously obtained [2] in the context of the four-dimensional gauged supergravity of Freedman and Schwarz [6]. With the substitution [2]

$$x = \rho^2\, e^{\xi(\rho)}\,, \qquad Y = -\rho\frac{d\xi(\rho)}{d\rho} - \rho^2\, e^{\xi(\rho)} - 1\,, \tag{4.7}$$

eq. (4.6) reduces to the Liouville equation

$$\frac{d^2\xi}{d\rho^2} = 2\,\mathrm{e}^{\xi}\,, \tag{4.8}$$

which is completely integrable. This leads to the following most general solution of the Bogomol'nyi equations that is regular at the origin of the spherical coordinate system:

$$ds^2 = r_0^2\,\mathrm{e}^{2\nu}\left\{dt^2 - d\rho^2 - Y\,d\Omega_2^2 - (dx^4)^2\right\}\,, \tag{4.9}$$

where r_0 is the integration constant and

$$Y = 2\rho\coth\rho - \frac{\rho^2}{\sinh^2\rho} - 1, \quad w = \frac{\rho}{\sinh\rho}\,, \qquad \mathrm{e}^{6\nu} = \frac{\sinh^2\rho}{Y}\,, \tag{4.10}$$

while the gauge filed and the dilaton are given by (4.2) and (4.3). Since $Y(\rho) = \rho^2 + O(\rho^4)$ for small ρ, the geometry is regular as $\rho \to 0$. The geometry is also globally regular, although, since $Y = 2\rho + O(1)$ as $\rho \to \infty$, the metric does not become flat for large ρ.

This five-dimensional solution is closely related to the solution of the gauged $D = 4$ supergravity of Freedman and Schwarz because the latter can be obtained via dimensional reduction plus truncation of the five-dimensional supergravity under consideration. In other words, the five dimensional solution can also be obtained by uplifting the four dimensional solution. The relation between the vielbeins in four and five dimensions is $\Theta^A = \mathrm{e}^{-\frac{1}{3}\tilde{\phi}}e^A$, where $A = 0,1,2,3$ and e^A is the D=4 tetrad, while $\Theta^4 = \mathrm{e}^{\frac{2}{3}\tilde{\phi}}dx^4$. The four dimensional dilaton, $\tilde{\phi}$, is related to the five dimensional one via $\phi = \sqrt{2/3}\,\tilde{\phi}$. The four-dimensional Yang-Mills field is obtained from the five-dimensional one by setting the fourth spacetime component to zero.

5. Concluding remarks

One can lift the above solutions to ten dimensions using the results of [7]. The bulk/boundary interpretation of the SO(3) solutions will then probably be similar to that for their D=4 counterparts [8] — they will provide the dual supergravity description for the NS 5-branes wrapped around S^2. It is less clear what the interpretation for the SO(4) solutions might be. Notice that these solutions do not have a simple asymptotic behavior — they do not approach the maximal (super)-symmetry backgrounds at infinity. This is due to the fact that we actually consider the half-gauged model, in which case the dilaton potential has no stationary points thus driving the dilaton asymptotically to infinity. Turning on the U(1) gauge coupling constant g_1 the potential becomes [11]

$$\mathrm{U}(\phi) = -\frac{1}{8}\exp\left(-2\sqrt{\frac{2}{3}}\phi\right) - \frac{g_1}{2\sqrt{2}}\exp\left(\sqrt{\frac{2}{3}}\phi\right), \tag{5.1}$$

and this does have a stationary point. This suggests that there could be asymptotically AdS solutions. In fact some of such solutions have recently been obtained [9]. The problem however is that unless $g_1 = 0$, the simple metric-dilaton relations as those in (3.6), (4.3) do not hold and there is no linear dependence between different components of the fermionic SUSY variations similar to the one in (3.15). This gives too many independent supersymmetry constraints, which will probably kill all supersymmetric solutions apart from the simplest ones (all solutions of [9] are simple in the sense that they belong to the embedded abelian type). However, a further analysis is required in order to make any definite statements.

References

[1] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *large-N field theories, string theory and gravity*, *Phys. Rep.* **323** (2000) 183 [hep-th/9905111].

[2] A.H. Chamseddine and M.S. Volkov, *Non-abelian BPS monopoles in $N = 4$ gauged supergravity*, *Phys. Rev. Lett.* **79** (1997) 3343 [hep-th/9707176].

[3] A.H. Chamseddine and M.S. Volkov, *Non-abelian solitons in $N = 4$ gauged supergravity and leading order string theory*, *Phys. Rev.* **D 57** (1998) 6242 [hep-th/9711181].

[4] G.W. Gibbons, D. Kastor, L.A.J. London, P.K. Townsend, and J. Traschen, *Supersymmetric self-gravitating solitons*, *Nucl. Phys.* **B 416** (1994) 850 [hep-th/9310118].

[5] G.W. Gibbons, P.K. Townsend, *Anti-gravitating monopoles and dyons*, *Phys. Lett.* **B 356** (1995) 472 [hep-th/9506131].

[6] D.Z. Freedman and J. Schwarz, *$N = 4$ supergravity model with local* $\mathrm{SU}(2) \times \mathrm{SU}(2)$ *invariance Nucl. Phys.* **B 137** (1978) 333

[7] H. Lu, C.N. Pope and T.A. Tran, *Five-dimensional $N = 4$,* $\mathrm{SU}(2) \times \mathrm{U}(1)$ *gauged supergravity from type IIB*, *Phys. Lett.* **B 475** (2000) 261 [hep-th/9909203].

[8] J.M. Maldacena and C. Nunez, *Towards the large-N limit of pure $N = 1$ super Yang-Mills*, *Phys. Rev. Lett.* **86** (2001) 588 [hep-th/0008001].

[9] H. Nieder and Y. Oz, *Supergravity and D-branes wrapping special lagrangian cycles*, *J. High Energy Phys.* **03** (2001) 008 [hep-th/0011288].

[10] L. Randall and R. Sundrum, *An alternative to compactification*, *Phys. Rev. Lett.* **83** (1999) 4690 [hep-th/9906064].

[11] L.J. Romans, *Gauged $N = 4$ supergravities in five dimensions and their magnetovac backgrounds*, *Nucl. Phys.* **B 267** (1986) 433.

[12] M.S. Volkov, *Euclidean Freedman-Schwarz model*, *Nucl. Phys.* **B 566** (2000) 173 [hep-th/9910116].

[13] M.S. Volkov and D. Maison, *Bogomolnyi equations for einstein-yang-mills-dilaton theory*, *Nucl. Phys.* **B 559** (1999) 591 [hep-th/9904174].

JHEP04(2001)023